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The Structure of p-fixed Autonilpotent Finite Groups

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The concept of autonilpotent groups was introduced by Moghaddam and Parvaneh in 2010 which is related to concepts of absolute centre and autocommutator subgroups that were first proposed by Hegarty in 1994 and 1997 respectively. In the present article we give some useful properties of such groups. In the first section, we define a subgroup GK_n for a given group G, as $\mathrm{GK}_n = \langle [g, \alpha^n] \mid g \in G,\ \alpha \in \mathrm{Aut}(G) \rangle$ which is characteristic subgroup of G. In this section, based on the induction method, we obtain the structure of GK_n as $\langle [g, \alpha]^n \mid g \in G, \alpha \in \mathrm{Aut}(G) \rangle$, $\langle [g, \alpha]^n [g, \alpha, \alpha]^{n(n-1)/2} \mid g \in G, \alpha \in \mathrm{Aut}(G) \rangle$ and $\langle (\prod_{i=0}^{n-1} [g, \alpha][g, \alpha, \alpha]^i)\ [g, \alpha, \alpha, \alpha]^{\frac{n(n-1)(n-2)}{6}} \mid g \in G, \alpha \in \mathrm{Aut}(G) \rangle$ in autonilpotent groups of classes 2, 3 and 4 respectively. In second section, n-fixed group is introduced as a group G with $\mathrm{GK}_n = 1$ for some $n \in \mathbb{N}$. Then among other results of n-fixed groups, we characterize some n-fixed groups. Based on the role of the absolute centre subgroup in the structure of the group, we prove a non-trivial finite group G is isomorphic to $\mathbb{Z}_2$ iff G is 1-fixed group also we show that abelian group G is p-fixed autonilpotent group iff it is isomorphic to $G \cong C_{2^k}$ where $1 \leq k \leq 3$.

INTRODUCTION

Let G be a group and $A = \mathrm{Aut}(G)$ denotes its full automorphism group. For $g \in G$, $\alpha \in A$, the element $[g, \alpha] = g^{-1}g^{\alpha}$ is an autocommutator of G. If α runs over to Inner automorphisms, then autocommutator is usual commutator. Concept of autocommutator and the subgroup that generated by it, as $K(G) = \langle [g, \alpha] \mid g \in G,\ \alpha \in \mathrm{Aut}(G) \rangle$ was introduced by Hegarty in 1997 (Hegarty 1997). Clearly if α is limited to inner automorphism then $K(G)$ is identical to derived subgroup. $K_n(G)$ is one extension of $K(G)$ which was defined by Parvaneh and Moghaddam in 2010 (Parvaneh & Moghaddam 2010, Kappe *et al.* 2017) as follows:

$$
\begin{aligned}
K_n(G) &= [K_{n-1}(G), \mathrm{Aut}(G)] \\
&= \langle [g, \alpha_1, \dots, \alpha_n] \mid g \in G, \alpha_i \in \mathrm{Aut}(G), \\
&\quad i = 1, \dots, n \rangle
\end{aligned}
$$

Furthermore, $K(G) \geq K_2(G) \geq \dots \geq K_n(G) \geq \dots$

represents lower autocentral series of $K_n(G)$.

Also Hegarty in 1994 introduced the concept of absolute centre of G, which is defined by L(G) ={g ∈ G; [g, α] = 1, ∀α ∈Aut(G)}. Clearly, L(G) is a characteristic subgroup and contained in the centre of G. This notion has been already studied in (Gholamian & Nasrabadi 2016, Kaboutari & Nasrabadi 2016, Alamshahi *et al.* 2022). Several studies have been conducted on the relationship between autocommutator and absolute centre and its effect on group classification (Haghparast *et al.* 2023). The concept of autonilpotent group is one of these mentioned effects.

According to $K_n(G)$ series, autonilpotent group defined as follows:

Definition 1.1

A group G is called autonilpotent of class c if $K_c(G) = 1$, for some $c \in \mathbb{N}$.
the smallest c with this property is said to be autonilpotency class of G.

The concept of autonilpotent group has been discussed and studied from different perspectives (Nasrabadi & Gholamian 2017).
In this article we introduce a new subgroup, say GK_n, and determine the structure of this subgroup in autonilpotent groups of class $c < 5$. Meanwhile introducing n-fixed groups, we classify some n-fixed autonilpotent groups.

METHOD

The results of this research, like other branches of pure sciences, are based on studying and following the works and studies of other researchers in this field. In fact, articles, books, thinking, pondering and focusing on unsolved issues and using reasoning based on knowledge is the general method of doing work. In this article, specifically, the results are obtained in two different parts. In the first part, GK_n subgroups are obtained for autonilpotent groups of class c<5, and the method of the proof is induction. An example is provided in each case for clarity. In the second part, n-fixed groups are classified according to the structure of GK_n subgroups. The results obtained in this section are based on the reasoning of the group structure and the previous findings of researchers in this field.

RESULTS AND DISCUSSION

In this part, we present the structure of the subgroup GK_n for the autonilpotent groups of class $c < 5$ in the form of theorems, and in each case, we present the corresponding example.

First we introduce new subgroup as follows:

$$GK_n = \langle\, [g, \alpha^n] \mid g \in G, \alpha \in Aut(G) \,\rangle$$

It is easily seen that GK_n is characteristic subgroup of the group G. Following theorems determine the structure of GK_n in autonilpotent groups of class $c < 5$. It is clear that in every autonilpotent group of class 1, GK_n is trivial subgroup.

Theorem 3.1

Let G be an autonilpotent group of class 2. Then

$$GK_n = \langle\, [g, \alpha]^n \mid g \in G, \alpha \in Aut(G) \,\rangle.$$

Proof:
We prove the theorem by induction on n. If $n = 1$, the theorem is true. Assume that the statement holds for $n = k$. Then we consider $n = k + 1$ to perform the last inductive step. By definition we have

$GK_{k+1} = \langle\, [g, \alpha^{k+1}] \mid g \in G, \alpha \in Aut(G) \rangle$.

But

$$\begin{aligned}[g, \alpha^{k+1}] &= g^{-1}\alpha^{k+1}(g) = \\ &= g^{-1}\alpha^{k}\left(gg^{-1}\alpha(g)\right) = \\ &= g^{-1}\alpha^{k}(g)\alpha^{k}[g, \alpha] = \\ &= [g, \alpha^{k}][g, \alpha][g, \alpha]^{-1}\alpha^{k}[g, \alpha] = \\ &= [g, \alpha^{k}]\,[g, \alpha]\,[g, \alpha, \alpha^{k}].\end{aligned}$$

Since G is autonilpotent of class 2, we have

$[g, \alpha, \alpha^k] = 1$ and by induction hypothesis $[g, \alpha^k] = [g, \alpha]^k$. Then

$$[g, \alpha^{k+1}] = [g, \alpha]^k [g, \alpha] = [g, \alpha]^{k+1}.$$

So the proof is completed.

Example 3.2

Table 1. Autonilpotent Group of Class 2

$\mathbb{Z}_4$	$\langle a \mid a^4 = 1 \rangle$
$Aut(\mathbb{Z}_4)$	$I: a \to a$ $\alpha: a \to a^3$
GK_n	$\begin{cases} 1\ , & if\ n\ is\ even \\ \langle a^2 \rangle, & if\ n\ is\ odd \end{cases}$

Theorem 3.3

Let G be an autonilpotent group of class 3. Then

$$\mathrm{GK}_n = \langle\, [\,\mathrm{g}\,,\alpha\,]^n\ [\mathrm{g},\alpha,\alpha]^{\mathrm{n(n-1)/2}} \big| \mathrm{g} \in\ \mathrm{G}, \alpha \in\ \mathrm{Aut(G)}\,\rangle.$$

Proof:

We prove the theorem by induction on n. If $\mathrm{n} = 1$, the theorem is true. Assume that the statement holds for $\mathrm{n} = \mathrm{k}$. Then we consider $\mathrm{n} = \mathrm{k} + 1$ to perform the last inductive step.

We have

$$[\mathrm{g}, \alpha^{k+1}] = [\mathrm{g}, \alpha]\alpha[\mathrm{g}, \alpha^k].$$

But induction hypothesis implies that

$$[\mathrm{g}, \alpha^{k+1}] =$$
$$= [\mathrm{g}, \alpha]\alpha\left([\mathrm{g}, \alpha]^k[\mathrm{g}, \alpha, \alpha]^{\frac{\mathrm{k(k-1)}}{2}}\right) =$$
$$= [\mathrm{g}, \alpha](\alpha[\mathrm{g}, \alpha])^k\left(\alpha[\mathrm{g}, \alpha, \alpha]^{\frac{\mathrm{k(k-1)}}{2}}\right). \quad (i)$$

Since G is autonilpotent of class three, we have

$$\alpha[\mathrm{g}, \alpha] = [\mathrm{g}, \alpha]\ [\mathrm{g}, \alpha, \alpha]$$

when $[\mathrm{g}, \alpha, \alpha]$ is commutative. Hence

$$(\alpha[\mathrm{g}, \alpha])^k = [\mathrm{g}, \alpha]^k[\mathrm{g}, \alpha, \alpha]^k \qquad (ii)$$

Moreover by autonilpotency of class 3

$$\alpha\,[\mathrm{g}, \alpha, \alpha] = [\mathrm{g}, \alpha, \alpha][\mathrm{g}, \alpha, \alpha, \alpha] = [\mathrm{g}, \alpha, \alpha]$$

Therefore

$$(\alpha[\mathrm{g}, \alpha, \alpha])^{\frac{\mathrm{k(k-1)}}{2}} = [\mathrm{g}, \alpha, \alpha]^{\frac{\mathrm{k(k-1)}}{2}} \qquad (iii)$$

(i), (ii) and (iii) result in

$$[\mathrm{g}, \alpha^{k+1}] =$$
$$= [\mathrm{g}, \alpha][\mathrm{g}, \alpha]^k[\mathrm{g}, \alpha, \alpha]^k[\mathrm{g}, \alpha, \alpha]^{\frac{\mathrm{k(k-1)}}{2}} =$$
$$= [\mathrm{g}, \alpha]^{k+1}[\mathrm{g}, \alpha, \alpha]^{\frac{\mathrm{k(k+1)}}{2}}$$

And the proof is completed.

Example 3.4

Table 2. Autonilpotent Group of Class 3

$\mathbb{Z}_8$	$\langle a \mid a^8 = 1\rangle$
$Aut(\mathbb{Z}_8)$	$I: a \to a$ $\alpha_2: a \to a^3$ $\alpha_3: a \to a^5$ $\alpha_4: a \to a^7$
GK_n	$\begin{cases} 1\ , & if\ n\ is\ even \\ \langle a^2\rangle, & if\ n\ is\ odd \end{cases}$

Theorem 3.5

Let G be an autonilpotent group of class 4. Then

$$\mathrm{GK}_n = \langle\ (\textstyle\prod_{i=0}^{n-1}[\mathrm{g},\alpha][\mathrm{g},\alpha,\alpha]^i)\ [\mathrm{g},\alpha,\alpha,\alpha]^{\frac{\mathrm{n(n-1)(n-2)}}{6}} \Big| \mathrm{g} \in \mathrm{G}, \alpha \in \mathrm{Aut(G)}\ \rangle.$$

Proof:

We prove the theorem by induction on n. If $\mathrm{n} = 1$, the theorem is true. Assume that the statement holds for $\mathrm{n} = \mathrm{k}$. Then we consider $\mathrm{n} = \mathrm{k} + 1$ to perform the last inductive step.

We have

$$[\mathrm{g},\alpha^{k+1}] =$$

$$[\mathrm{g},\alpha]\alpha(\textstyle\prod_{i=0}^{k-1}[\mathrm{g},\alpha][\mathrm{g},\alpha,\alpha]^i)\ [\mathrm{g},\alpha,\alpha,\alpha]^{\frac{\mathrm{k(k-1)(k-2)}}{6}} =$$

$$= (\prod_{i=0}^{k}[\mathrm{g},\alpha][\mathrm{g},\alpha,\alpha]^i)\ [\mathrm{g},\alpha,\alpha,\alpha]^{\frac{\mathrm{k(k-1)(k-2)}}{6}}$$

$$[\mathrm{g},\alpha,\alpha,\alpha]^{\frac{\mathrm{k(k-1)(k-2)}}{6}+\sum_{i=0}^{k-1} i}$$

$$= (\prod_{i=0}^{k}[\mathrm{g},\alpha][\mathrm{g},\alpha,\alpha]^i)\ [\mathrm{g},\alpha,\alpha,\alpha]^{\frac{\mathrm{k(k-1)(k-2)}}{6}+\frac{\mathrm{k(k-1)}}{2}}$$

$$= (\prod_{i=0}^{(k+1)-1}[\mathrm{g},\alpha][\mathrm{g},\alpha,\alpha]^i)\ [\mathrm{g},\alpha,\alpha,\alpha]^{\frac{\mathrm{k(k+1)(k-1)}}{6}}$$

Thus the proof is completed.

Example 3.6

Table 3. Autonilpotent Group of Class 4

$\mathbb{Z}_{16}$	$\langle a \mid a^{16} = 1\rangle$
$Aut(\mathbb{Z}_{16})$	$I: a \to a$ $\alpha_2: a \to a^3$ $\alpha_3: a \to a^5$ $\alpha_4: a \to a^7$ $\alpha_5: a \to a^9$ $\alpha_6: a \to a^{11}$ $\alpha_7: a \to a^{13}$ $\alpha_8: a \to a^{15}$
GK_n	$\begin{cases} 1, & n = 0(mod4) \\ \langle a^2\rangle, & n = 1 or\ 3(mod4) \\ \langle a^8\rangle, & n = 2(mod4) \end{cases}$

Relation between autonilpotent and n-fixed groups

In this section first n-fixed groups are introduced. Then the structure of some n-fixed autonilpotent groups is determined.

Definition 3.7

A group G is called n-fixed if $\mathrm{GK}_n = 1$. Similarly to autonilpotency class, a class for n-fixed group is the smallest number, n, of feature $\mathrm{GK}_n = 1$.

Theorem 3.8

If G be a finite group with elementary abelian automorphism group of exponent p,then G is p –fixed group.

Proof:

The result is simply obtained by definition of n-fixed group.

Example 3.9

One can construct a non-abelian group G of order 64 such that Aut(G) is an elementary abelian group of order 128 which is the smallest p-fixed p-group (Bolinches *et al.* 2023).

D. Jonah and M. Konvisser constructed 4-generated groups of order p^8 such that Aut(G) is an elementary abelian group of order p^{16}, where p is any prime which is the p-fixed p-group(Jonah & Konvisser 1975).

Also more non-abelian n-fixed group examples were studied by other researchers. (Munia et al. 2022, Hosseini et al. 2025, Ghoraishi 2019).

Lemma 3.10 ((Hosseini et al. 2015), Lemma 2.9.)

If G is an autonilpotent group, then every Sylow p-subgroup of G is also autonilpotent.

Lemma 3.11 ((Hosseini et al. 2015), Lemma 2.2.)

If G is a finite p-group of autonilpotency class c, then $\mathrm{Aut(G)}$ is also a p-group.

Lemma 3.12

Let G be n-fixed group such that $\mathrm{G = H \times K}$ with $\mathrm{H, K \leq G}$ and $(|\mathrm{H}|, |\mathrm{K}|) = 1$. Then H and K are n-fixed.

Proof:

Without loss of generality we may assume that

$\mathrm{HK}_n \neq 1$. So there exist $\mathrm{h \in H}$ and $\alpha_H \in \mathrm{Aut(H)}$ such that $[\mathrm{h}, \alpha_H^n] \neq 1$. Thus for $\mathrm{g} = (\mathrm{h}, \mathrm{e}) \in \mathrm{H \times K = G}$ and $\alpha = \alpha_H \times \mathrm{I}_K \in \mathrm{Aut(H) \times Aut(K) \cong Aut(G)}$, we have $[\mathrm{g}, \alpha^n] \neq 1$ which is in contradiction to $\mathrm{GK}_n = 1$ and the proof is completed.

Following theorems classify some n-fixed autonilpotent groups.

Theorem 3.13

G is an abelian, p-fixed autonilpotent group (prime p) if and only if $\mathrm{G \cong C}_{2^k}$ where $1 \leq \mathrm{k} \leq 3$.

Proof:

Clearly, if the group G is considered to be C_{2^k} where $1 \leq \mathrm{k} \leq 3$ the result is obvious. Conversely, we can write

$$\mathrm{G \cong S_1 \times S_2 \times \ldots \times S}_t$$

where S_i is an abelian Sylow p_i-subgroup of G ($1 \leq \mathrm{i} \leq \mathrm{t}$). Moreover S_i is autonilpotent and p-fixed by Lemma 3.8 and Lemma 3.10 respectively. Therefore without loss of generality we focus on the study of S_j as arbitrary Sylow subgroup of G .Clearly S_j is an abelian p -fixed autonilpotent p_j-group. Since S_j is p -fixed group, we have $[\mathrm{g}, \alpha^p] = 1$ for all. $\mathrm{g} \in \mathrm{S}_j$ and $\alpha \in \mathrm{Aut}(\mathrm{S}_j)$. Thus $\mathrm{Aut}(\mathrm{S}_j)$ is elementary abelian p -group. Hence S_j is cyclic and p -group. So, it is easily concluded that p =2 and $\mathrm{G \cong C}_{2^k}$ where $1 \leq \mathrm{k} \leq 3$.

Theorem 3.14

Let G be a non-abelian, autonilpotent group. Then G is p -fixed (prime p) if and only if G is p-group (prime p) with p-automorphism group of exponent p.

Proof:

Clearly, if the group G is considered to be p-group (prime p) with p-automorphism group of exponent p, the result is obvious. Conversely, assume the contrary and there exists a prime $q(q \neq p)$ such that $q \| G|$. Hence according to the structure of G and $\mathrm{Aut}(G)$ which was discussed in detail in the proof of the previous theorem, G must be n-fixed where $n = pq$ which contradicts that the group G is being a p–fixed group. Moreover since G is autonilpotent, the automorphism of G is also p-group by Lemma 3.11 and because G is p–fixed, the exponent of $\mathrm{Aut}(G)$ equals p and the proof is completed.

The following technical theorem is needed in proving one of our main results.

Theorem 3.15

If G is a finite abelian group, then the absolute centre of G is either trivial or $\mathbb{Z}_2$, the cyclic group of order 2.

Proof:

Clearly, in an abelian group G, the map $\theta: G \to G$ given by $\theta(x) = x^{-1}$ for all $x \in G$ is an automorphism. By the definition every element $g \in L(G)$ is fixed by all the automorphisms of G and so $g = \theta(g) = g^{-1}$. Hence all non-trivial elements g in $L(G)$ are of orders 2. Therefore the absolute centre of G is an abelian 2-group. Hence if G is abelian of odd order it has no involutions, which implies that $L(G)$ must be trivial. So we assume that G is an abelian group of even order and determine the structure of $L(G)$.

Thus we may consider $G = H \times K$, where H is a 2-group and K is a subgroup of odd order. If $\exp(H) = 2^n$, then H contains an element x, say of order 2^n and we may write $H = \langle x \rangle \times A$, where A is a subgroup of H of exponent less than or equal to 2^n. Now, every element g in G of order 2 must be of the following form:

$$g = x^{2^{n-1}}a, \qquad a \text{ or } x^{2^{n-1}}, \text{for some } a \in A.$$

Now, we show that the elements in $L(G)$ are only of the form $x^{2^{n-1}}$. If $g = x^{2^{n-1}}a$ or g=a, then G has an automorphism α , which sends a to $x^{2^{n-1}}a$. Therefore α moves both elements a and $x^{2^{n-1}}a$, and so they can not be in $L(G)$ and so g must be the form $x^{2^{n-1}}$. Now, to complete the proof we consider two cases for $\exp(A)$.

Case 1: $\exp(A) = 2^n$

In this case there exists an element $y \in A$ such that $|y| = 2^n$. As above, we may define an automorphism β such that $\beta(x) = xy$. Hence $\beta(g) = gy^{2^{n-1}}$ and so $\notin L(G)$. Thus in this case, the absolute centre is trivial.

Case 2: $\exp(A) < 2^n$

Assume that α is an arbitrary automorphism, then $\alpha(x) = x^{2k-1}y$, where $y \in A$ and $|y| < |x|$. Hence $\alpha(g) = \alpha\left(x^{2^{n-1}}\right) = (x^{2k-1}y)^{2^{n-1}} = g$,which shows that $g \in L(G)$. Since g is of order 2, it implies that $L(G) \cong \mathbb{Z}_2$.

Theorem 3.16

Let G be a non-trivial finite group, then $G \cong \mathbb{Z}_2$ if and only if G is 1-fixed.

Proof:

It is clear that $\mathbb{Z}_2$ is 1-fixed. Conversely, assume that G is 1-fixed group. So we consider two cases.

Case 1: G is abelian

The autocommutator subgroup of G is trivial. therefore $G = L(G)$. By Theorem 3.13 $G \cong \mathbb{Z}_2$.

Case 1: G is non-abelian

It is clear that automorphism group of G has non-identity element, say, α. Abviously, there exists $x \epsilon G$ such that $[x, \alpha] \neq 1$. Therefore G cannot be 1-fixed group.

CONCLUSION

Group theory is an interesting area of research. One of the most important goals of researchers is to find the structure of finite groups because it makes the study of groups easier. This work relates the finite autonilpotent group structures. In this paper, the characteristic subgroup, 〖GK〗_n, is introduced and while determining the exact structure of this subgroup in autonilpotent groups of class $c<5$, it has been used as a tool in defining n-fixed groups. Also finite autonilpotent groups are classified when their above mentioned characteristic subgroup is the trivial subgroup.

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